

# On the value-distribution of the difference between logarithms of two symmetric power $L$ -functions

Kohji Matsumoto\* and Yumiko Umegaki†

## Abstract

We consider the value distribution of the difference between logarithms of two symmetric power  $L$ -functions at  $s = \sigma > 1/2$ . We prove that certain averages of those values can be written as integrals involving a density function which is constructed explicitly.

## 1 Introduction and the statement of main results.

Let  $f$  be a primitive form of weight  $k$  and level  $N$ , which means that it is a normalized common Hecke eigen new form of weight  $k$  for  $\Gamma_0(N)$ . We denote by  $S_k(N)$  the set of all cusp forms of weight  $k$  and level  $N$ . Any  $f \in S_k(N)$  has a Fourier expansion at infinity of the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}, \quad \lambda_f(1) = 1.$$

In the case  $f$  is a normalized common Hecke eigen form, the Fourier coefficients  $\lambda_f(n)$  are real numbers. We consider the  $L$ -function

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

associated with a primitive form  $f$  where  $s = \sigma + i\tau \in \mathbb{C}$ . This is absolutely convergent when  $\sigma > 1$ , but can be continued to the whole of  $\mathbb{C}$  as an entire function.

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\*The first author was supported by JSPS Grant-in-Aid for Scientific Research (B) Grant Number 25287002

†The second author was supported by JSPS Grant-in-Aid for Young Scientists (B) Grant Number 23740020.

We denote by  $\mathbb{P}$  the set of all prime numbers. We know that  $L(f, s)$  has the Euler product

$$\begin{aligned} L(f, s) &= \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1} \\ &= \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} (1 - \alpha_f(p)p^{-s})^{-1} (1 - \beta_f(p)p^{-s})^{-1}, \end{aligned}$$

where  $\beta_f(p)$  is the complex conjugate of  $\alpha_f(p)$ . Note that  $\alpha_f(p)$  and  $\beta_f(p)$  satisfy  $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$  and  $|\alpha_f(p)| = |\beta_f(p)| = 1$ . This Euler product is deduced from the relations

$$\lambda_f(p^\ell) = \begin{cases} \lambda_f^\ell(p) & p \mid N, \\ \sum_{h=0}^{\ell} \alpha_f^{\ell-h}(p) \beta_f^h(p) & p \nmid N. \end{cases} \quad (1)$$

In the present paper, we consider the value of  $\log L(\text{Sym}_f^\mu, s) - \log L(\text{Sym}_f^\nu, s)$  at  $s = \sigma > 1/2$ , where the  $\gamma$ -th symmetric power  $L$ -function is defined by

$$\begin{aligned} L(\text{Sym}_f^\gamma, s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n^\gamma)}{n^s} \\ &= \prod_{\substack{p \in \mathbb{P} \\ p|N}} (1 - \lambda_f(p^\gamma)p^{-s})^{-1} \prod_{\substack{p \in \mathbb{P} \\ p \nmid N}} \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1} \end{aligned}$$

for  $\sigma > 1$ . In the case  $\gamma = 1$ , clearly  $L(\text{Sym}_f^1, s) = L(f, s)$ . In general, it is believed that the symmetric power  $L$ -function could be continued to an entire function and would satisfy a functional equation. We suppose the analytic continuation and its holomorphy for  $\sigma > 1/2$  in Assumption 1 below.

For  $\sigma > 1$ , we define

$$\log L(\text{Sym}_f^\gamma, s) = - \sum_{\substack{p \in \mathbb{P} \\ p|N}} \text{Log}(1 - \lambda_f(p^\gamma)p^{-s}) - \sum_{\substack{p \in \mathbb{P} \\ p \nmid N}} \sum_{h=0}^{\gamma} \text{Log}(1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}),$$

where  $\text{Log}$  means the principal branch. In the strip  $1/2 < \sigma \leq 1$ , we suppose it can be analytically continued to  $\sigma > 1/2$  under Assumption 2 below, which claims that  $L(\text{Sym}_f^\gamma, s)$  has no zero in the strip  $1/2 < \sigma \leq 1$ . In this paper we introduce the following two assumptions.

**Assumption 1.** Let  $f$  be a primitive form of even weight  $k$  where  $2 \leq k < 12$  or  $k = 14$ . The level of  $f$  is  $q^m$ , where  $q$  is a prime number. For a fixed positive integer  $\gamma$ , the symmetric power  $L$ -function  $L(\text{Sym}_f^\gamma, s)$  is analytically continued to a holomorphic function in  $\sigma > 1/2$ . Moreover it satisfies the estimate

$$|L(\text{Sym}_f^\gamma, s)| \ll_\gamma q^m (|\tau| + 2) \quad (2)$$

for  $1/2 < \sigma \leq 2$ .

**Remark 1.** For the symmetric power  $L$ -function, if we obtain a suitable functional equation which is the same type as in Cogdell and Michel [2], we have the same estimate as (2) by using the Phlagmén-Lindelöf principle. As Cogdell and Michel mentioned in [2], this assumption is held in the case when  $f$  is a primitive form of weight 2 and of square-free level for the symmetric cube  $L$ -function, which is proved by Kim and Shahidi [13].

**Remark 2.** For a primitive form of weight  $k$  and level  $M$ , where  $k$  is an even positive integer and  $M$  is a positive integer, the automorphic  $L$ -function  $L(f, s)$  is entire and it has a functional equation. The estimate of the form (2) holds for  $L(f, s)$ , that is

$$|L(\text{Sym}_f^1, s)| = |L(f, s)| \ll M(|\tau| + 2) \quad (3)$$

for  $1/2 < \sigma \leq 2$ .

**Assumption 2.** Let  $f$  be a primitive form of weight  $k$  which  $2 \leq k < 12$  or  $k = 14$ . The level is  $q^m$ , where  $q$  is a prime number. For a fixed positive integer  $\gamma$ , the  $L$ -functions  $L(\text{Sym}_f^\gamma, s)$  satisfies Generalized Riemann Hypothesis (GRH) which means that  $L(\text{Sym}_f^\gamma, s)$  has no zero in the strip  $1/2 < \sigma \leq 1$ .

In this paper, we mainly consider two types of averages which are defined below. For the definitions of them, we first prepare the notations. Let  $q$  be a prime number. For any series  $\{A_f\}$  over primitive forms  $f \in S_k(q^m)$ , where  $2 \leq k < 12$  or  $k = 14$ , we use the symbol  $\sum'$  in the following sense:

$$\sum'_{f \in S_k(q^m)} A_f = \frac{1}{C_k(1 - C_q(m))} \sum_{\substack{f \in S_k(q^m) \\ f: \text{primitive form}}} \frac{A_f}{\langle f, f \rangle},$$

where  $C_k$  and  $C_q(m)$  are constants defined by

$$C_k = \frac{(4\pi)^{k-1}}{\Gamma(k-1)}, \quad C_q(m) = \begin{cases} 0 & m = 1, \\ q(q^2 - 1)^{-1} & m = 2, \\ q^{-1} & m \geq 3. \end{cases}$$

These constants appeared in Lemma 3 in the second author [4] (see (7) below), which came from Petersson's formula.

We define the “partial” Euler product of the symmetric power  $L$ -function by

$$L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) = \prod_{p \in \mathbb{P}(q)} \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1}$$

for a primitive form  $f$  of level  $q^m$ , where  $q$  is a prime number and the subset  $\mathbb{P}(q) \subset \mathbb{P}$  means the set of all prime numbers except for the fixed prime number  $q$ . Let  $\mu > \nu \geq 1$  be integers with  $\mu - \nu = 2$ . By  $Q(\mu)$  we denote the smallest

prime number satisfying  $2^\mu/\sqrt{Q(\mu)} < 1$ . In this paper, we study two types of averages which are defined by

$$\begin{aligned} & \text{Avg}_{\text{prime}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{\substack{q \rightarrow \infty \\ q: \text{prime} \\ m: \text{fixed}}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \end{aligned} \quad (4)$$

and

$$\begin{aligned} & \text{Avg}_{\text{power}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{\substack{m \rightarrow \infty \\ q: \text{fixed prime} \\ (\text{If } 1 \geq \sigma > 1/2, q \geq Q(\mu))}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned} \quad (5)$$

where  $\Psi$  is a  $\mathbb{C}$ -valued function defined on  $\mathbb{R}$ . On the above average  $\text{Avg}_{\text{power}}$ , we consider  $q \geq Q(\mu)$  when  $1 \geq \sigma > 1/2$ . The reason is technical which will be mentioned in Section 5. The main theorem in the present paper is as follows.

**Theorem 1.** *Let  $\mu > \nu \geq 1$  be integers with  $\mu - \nu = 2$ . Suppose Assumptions 1 and 2 when  $\gamma$  is  $\mu$  and  $\nu$ . Let  $k$  be an even integer which satisfies  $2 \leq k < 12$  or  $k = 14$ . Then, for  $\sigma > 1/2$ , there exists a function  $\mathcal{M}_\sigma : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  which can be explicitly constructed, and for which the formula*

$$\begin{aligned} & \text{Avg}_{\text{prime}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{power}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \end{aligned}$$

holds for any  $\Psi : \mathbb{R} \rightarrow \mathbb{C}$  which is a bounded continuous function or a compactly supported characteristic function.

The above restriction on the weight  $k$  is necessary to prove (7) below.

We mention a corollary of the  $\text{Avg}_{\text{prime}}$  part of the theorem. Consider the following different type of averages, involving summations with respect to levels:

$$\begin{aligned} & \text{Avg}_{\text{primesum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{X \rightarrow \infty} \frac{1}{\pi(X)} \sum_{\substack{q \leq X \\ q: \text{prime} \\ m: \text{fixed}}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned}$$

where  $\pi(X)$  denotes the number of prime numbers not larger than  $X$ , and

$$\begin{aligned} & \text{Avg}_{\text{primepowersum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \lim_{X \rightarrow \infty} \frac{1}{\pi^*(X)} \sum_{\substack{q^m \leq X \\ q: \text{prime} \\ m \geq 1}} \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)), \end{aligned}$$

where  $\pi^*(X)$  denotes the number of all pairs  $(q, m)$  of a prime number  $q$  and a positive integer  $m$  with  $q^m \leq X$ .

**Corollary 1.** *Under the same assumptions as Theorem 1, we have*

$$\begin{aligned} & \text{Avg}_{\text{primesum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{primepowersum}} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned}$$

**Remark 3.** Theorem 1 can be generalized to the case of any average defined by some limit, which is different from those in (4) and (5), but satisfies the condition that  $q^m \rightarrow \infty$ . (For example,  $q \rightarrow \infty$  with  $m = m(q)$  moving arbitrarily.) In fact, from the proof we can see that the only necessary limit procedure is  $q^m \rightarrow \infty$ .

The first result of this type is due to Bohr and Jessen [1]. Let  $\zeta(s)$  be the Riemann zeta-function. Bohr and Jessen proved that, when  $\Psi$  is a compactly supported characteristic function defined on  $\mathbb{C}$ , the formula

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \Psi(\log \zeta(s + i\tau)) d\tau = \int_{\mathbb{C}} \mathcal{M}_{\zeta, \sigma}(w) \Psi(w) \frac{dudv}{\sqrt{2\pi}} \quad (6)$$

holds for  $\Re s > 1/2$  (where  $w = u + iv$ ), with a certain density function  $\mathcal{M}_{\zeta, \sigma}$ . The analogue for the logarithmic derivative  $\zeta'/\zeta(s)$  was first proved by Kershner and Wintner [12].

Ihara [5] discovered that the same type of results can be shown for certain mean values of  $L'/L(s, \chi)$  with respect to characters, where  $L(s, \chi)$  denotes the Dirichlet (or Hecke)  $L$ -function attached to the character  $\chi$ , including also the function field case. Ihara's work was strengthened, and extended to the  $\log L$  case, in several joint papers of Ihara and the first author [6], [7], [8], [9]. Recently, Mourtada and Murty [14] obtained an analogous result for the mean value of  $L'/L(s, \chi)$  with respect to discriminants.

In those former results, the function  $\Psi$  is defined on  $\mathbb{C}$ , and the right-hand side of the formula is an integral over  $\mathbb{C}$ . However in our Theorem 1, the function  $\Psi$  is defined on  $\mathbb{R}$ , and the right-hand side is an integral over  $\mathbb{R}$ . This is one remarkable difference of our present work from the former researches.

The plan of this paper is as follows. Section 2 is the preparation, with the proof of Corollary 1. In Section 3 we construct the density function  $\mathcal{M}_\sigma$ , in Section 4 we state the key lemma (Lemma 2) and prove it in the case  $\sigma > 1$ , in Section 5 we prepare certain approximation of  $L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$  to prove the key lemma, in Section 6 we prove the key lemma for  $1 \geq \sigma > 1/2$  and finally, in Section 7 we will complete the proof of Theorem 1. The basic structure of our argument is similar to the previous work by Ihara and the first author [7].

**Remark 4.** It is surely interesting to search for the density function, of the nature similar to the above, for the average of  $\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$  itself for  $f \in$

$S_k(q^m)$ . However to obtain such a density function is difficult by our present method. The reason is explained in Remark 5 below. Therefore we consider the difference between the logarithm of symmetric  $\mu$ -th power  $L$  function and that of symmetric  $\nu$ -th power  $L$  function where  $\mu$  and  $\nu$  are of the same parity. However we have another difficulty in the case  $\mu - \nu > 2$ . It is explained in Remark 6 below. Hence Theorem 1 is shown only in the case  $\mu - \nu = 2$ .

**Acknowledgment.** The authors would like to express their gratitude to Professor Masaaki Furusawa and Professor Atsuki Umegaki for their valuable comments.

## 2 Preparations.

Bohr and Jessen [1] used the Kronecker-Weyl theorem on uniform distribution of sequences as an essential tool in the proof of (6). In Ihara [5], the corresponding tool is the orthogonality relation of characters.

In our present situation, the corresponding useful tool is Petersson's well-known formula (see, e.g., [10]). In the proof of our main Theorem 1, we will use the following formula ((7) below) for a prime number  $q$ , which was shown in Lemma 3 in the second author [4]. This formula embodies the essence of Petersson's formula, in the form suitable for our present aim.

When  $2 \leq k < 12$  or  $k = 14$ , for the primitive form  $f$  of weight  $k$  and level  $q^m$ , we have

$$\sum'_{f \in S_k(q^m)} \lambda_f(n) = \delta_{1,n} + \begin{cases} O_k(n^{(k-1)/2} q^{-k+1/2}) & m = 1, \\ O_k(n^{(k-1)/2} q^{m(-k+1/2)} q^{k-3/2}) & m \geq 2, \end{cases} \quad (7)$$

where  $\delta_{1,n} = 1$  if  $n = 1$  and 0 otherwise. We denote the error term in (7) by  $n^{(k-1)/2} E(q^m)$ , that is

$$\sum'_{f \in S_k(q^m)} \lambda_f(n) - \delta_{1,n} = n^{(k-1)/2} E(q^m).$$

Then we have

$$E(q^m) \ll q^{-k+1/2} \quad (8)$$

for any  $m$ , and

$$E(q^m) \ll \begin{cases} q^{-3/2} & m = 1, \\ q^{-5/2} & m = 2, \\ q^{-1-m} & m \geq 3, \end{cases} \ll q^{-m} \quad (9)$$

for any  $m$ . Also in the case  $n = 1$ , the formula (7) implies

$$\sum'_{f \in S_k(q^m)} \lambda_f(1) = \sum'_{f \in S_k(q^m)} 1 = 1 + E(q^m) \ll 1. \quad (10)$$

Let  $\mathcal{P}$  be a subset of  $\mathbb{P}$  and  $q$  a fixed prime number. For a primitive form  $f$  of weight  $k$  and level  $q^m$ , define

$$L_{\mathcal{P}}(\text{Sym}_f^\gamma, s) = \prod_{p \in \mathcal{P}} L_p(\text{Sym}_f^\gamma, s),$$

where

$$L_p(\text{Sym}_f^\gamma, s) = \begin{cases} (1 - \lambda_f(p^\gamma) p^{-s})^{-1} & p = q, \\ \prod_{h=0}^{\gamma} (1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s})^{-1} & p \neq q. \end{cases}$$

Especially we write

$$L_{\mathcal{P}}(f, s) = L_{\mathcal{P}}(\text{Sym}_f^1, s).$$

Further, for integers  $\mu > \nu > 0$  with  $\mu - \nu = 2$ , we put

$$L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s) = \frac{L_{\mathcal{P}}(\text{Sym}_f^\mu, s)}{L_{\mathcal{P}}(\text{Sym}_f^\nu, s)}.$$

This can be defined for  $\sigma > 1/2$  under Assumption 2.

Now let  $\mathcal{P}$  be a finite subset of  $\mathbb{P}(q)$ . We define the topological group  $\mathcal{T}_{\mathcal{P}}$  by

$$\mathcal{T}_{\mathcal{P}} = \prod_{p \in \mathcal{P}} \mathcal{T},$$

where  $\mathcal{T} = \{t \in \mathbb{C} \mid |t| = 1\}$ . For a fixed  $\sigma > 1/2$ , we consider the function

$$g_{\sigma, \mathcal{P}}(t_{\mathcal{P}}) = \sum_{p \in \mathcal{P}} g_{\sigma, p}(t_p)$$

on  $t_{\mathcal{P}} = (t_p)_{p \in \mathcal{P}} \in \mathcal{T}_{\mathcal{P}}$ , where

$$g_{\sigma, p}(t_p) = -\log(1 - t_p p^{-\sigma}).$$

For any  $s = \sigma + i\tau$  ( $\sigma > 1/2$ ) we have

$$\begin{aligned} & \log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s) \\ &= \log L_{\mathcal{P}}(\text{Sym}_f^\mu, s) - \log L_{\mathcal{P}}(\text{Sym}_f^\nu, s) \\ &= \sum_{p \in \mathcal{P}} \left( -\sum_{h=0}^{\mu} \log(1 - \alpha_f^{\mu-h}(p) \beta_f^h(p) p^{-s}) + \sum_{h=0}^{\nu} \log(1 - \alpha_f^{\nu-h}(p) \beta_f^h(p) p^{-s}) \right) \\ &= \sum_{p \in \mathcal{P}} \left( -\log(1 - \alpha_f^\mu(p) p^{-s}) - \log(1 - \beta_f^\nu(p) p^{-s}) \right) \\ &= \sum_{p \in \mathcal{P}} \left( g_{\sigma, p}(\alpha_f^\mu(p) p^{-i\tau}) + g_{\sigma, p}(\beta_f^\nu(p) p^{-i\tau}) \right) \\ &= g_{\sigma, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}) \mathcal{P}^{-i\tau}) + g_{\sigma, \mathcal{P}}(\beta_f^\nu(\mathcal{P}) \mathcal{P}^{-i\tau}), \end{aligned} \tag{11}$$

where  $\alpha_f^\mu(\mathcal{P}) = (\alpha_f^\mu(p))_{p \in \mathcal{P}}$ ,  $\beta_f^\mu(\mathcal{P}) = (\beta_f^\mu(p))_{p \in \mathcal{P}}$  and  $\mathcal{P}^{-i\tau} = (p^{-i\tau})_{p \in \mathcal{P}}$ . In the above equation, we used the fact that  $\beta_f(p)$  is the complex conjugate of  $\alpha_f(p)$ , and hence  $\alpha_f^{\mu-h}(p)\beta_f^h(p) = \alpha_f^{\nu-(h-1)}(p)\beta_f^{h-1}(p)$  ( $1 \leq h \leq \mu-1$ ). Hereafter, we sometimes write  $\alpha_f(p) = e^{i\theta_f(p)}$  and  $\beta_f(p) = e^{-i\theta_f(p)}$ .

In the case  $\sigma > 1$ , we deal with the value  $L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, \sigma + i\tau)$  as the limit of the value  $L_{\mathcal{P}}(\text{Sym}_f^\gamma, \sigma + i\tau)$  as  $\mathcal{P}$  tends to  $\mathbb{P}(q)$ . In fact, from (11) we have

$$\begin{aligned} \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma + i\tau) \\ = \lim_{\substack{\mathcal{P} \rightarrow \mathbb{P}(q) \\ \mathcal{P} \subset \mathbb{P}(q)}} (g_{\sigma, \mathcal{P}}(\alpha_f^\mu(\mathcal{P})\mathcal{P}^{-i\tau}) + g_{\sigma, \mathcal{P}}(\beta_f^\mu(\mathcal{P})\mathcal{P}^{-i\tau})). \end{aligned} \quad (12)$$

In the case  $1 \geq \sigma > 1/2$ , we will prove the relation between  $\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, \sigma)$  and  $\log L_{\mathcal{P}}(\text{Sym}_f^\gamma, \sigma)$  with a suitable finite subset  $\mathcal{P} \subset \mathbb{P}(q)$  depending on  $q^m$  and will consider the averages of them. This will be given in Lemma 3 in Section 5.

Now we conclude this section with the proof of Corollary 1.

*Proof of Corollary 1.* Write

$$A(q^m) = \sum'_{f \in S_k(q^m)} \Psi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)).$$

The  $\text{Avg}_{\text{prime}}$  part of the theorem implies that, for any  $\varepsilon > 0$ , there exists a  $Q_0 = Q_0(\varepsilon)$  for each fixed  $m$  such that

$$\left| A(q^m) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \right| < \varepsilon$$

for any prime  $q > Q_0$ . This clearly implies the first part of the corollary. As for the second part, first note that  $\pi^*(X)$  in the denominator can be replaced by  $\pi(X)$ , because  $\lim_{X \rightarrow \infty} \pi(X)/\pi^*(X) = 1$ . We divide the sum as

$$\begin{aligned} \frac{1}{\pi(X)} \sum_{\substack{q^m \leq X \\ q: \text{prime} \\ m \geq 1}} A(q^m) \\ = \frac{1}{\pi(X)} \sum_{q \leq X} A(q) + \frac{1}{\pi(X)} \sum_{2 \leq m \leq \lfloor \log X / \log 2 \rfloor} \sum_{q \leq X^{1/m}} A(q^m). \end{aligned} \quad (13)$$

Using (10) and the fact that  $\Psi$  is bounded, we find that  $A(q^m)$  is bounded. Hence the second term on the right-hand side of (13) is

$$\ll \frac{1}{\pi(X)} \sum_m \pi(X^{1/m}) \leq \frac{1}{\pi(X)} \sum_m \pi(X^{1/2}) \ll X^{-1/2} \log X,$$

which tends to 0 as  $X \rightarrow \infty$ . Lastly we apply the case  $m = 1$  of the first part of the corollary to the first term on the right-hand side of (13) to find that it tends to the desired integral.  $\square$



### 3 The density function $\mathcal{M}_\sigma$ .

Now we start the proof of our main theorem. In this section we first construct the density function  $\mathcal{M}_{\sigma, \mathcal{P}}$  for a finite set  $\mathcal{P} \subset \mathbb{P}(q)$ . By  $|\mathcal{P}|$  we denote the number of the elements of  $\mathcal{P}$ .

**Proposition 1.** *For any  $\sigma > 0$ , there exists a non-negative function  $\mathcal{M}_{\sigma, \mathcal{P}}$  defined on  $\mathbb{R}$  which satisfies following two properties.*

- *The support of  $\mathcal{M}_{\sigma, \mathcal{P}}$  is compact.*
- *For any continuous function  $\Psi$  on  $\mathbb{R}$ , we have*

$$\int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \Psi(u) \frac{du}{\sqrt{2\pi}} = \int_{\mathcal{T}_{\mathcal{P}}} \Psi(2\Re(g_{\sigma, \mathcal{P}}(t_{\mathcal{P}}))) d^*t_{\mathcal{P}},$$

where  $d^*t_{\mathcal{P}}$  is the normalized Haar measure of  $\mathcal{T}_{\mathcal{P}}$ . In particular, taking  $\Psi \equiv 1$ , we have  $\int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \frac{du}{\sqrt{2\pi}} = 1$ .

*Proof.* We construct the function  $\mathcal{M}_{\sigma, \mathcal{P}}$  by using the method similar to that in Ihara and the first author [7].

In the case  $|\mathcal{P}| = 1$  namely  $\mathcal{P} = \{p\}$ , we define a one-to-one correspondence from the open set  $(-\pi, 0)$  to its image  $A(\sigma, p) \subset \mathbb{R}$  by

$$u = u(\theta) = -2 \log |1 - e^{i\theta} p^{-\sigma}|$$

for  $\theta \in (-\pi, 0)$ . In fact, since

$$\frac{du}{d\theta} = -\frac{2p^{-\sigma} \sin(\theta)}{|1 - e^{i\theta} p^{-\sigma}|^2}, \quad (14)$$

we see that  $u$  is monotonically increasing with respect to  $\theta$ , hence one to one. The definition of  $\mathcal{M}_{\sigma, \mathcal{P}} = \mathcal{M}_{\sigma, p}$  is

$$\mathcal{M}_{\sigma, p}(u) = \begin{cases} \frac{|1 - e^{i\theta} p^{-\sigma}|^2}{-\sqrt{2\pi} \sin(\theta) p^{-\sigma}} & u \in A(\sigma, p), \\ 0 & \text{otherwise.} \end{cases}$$

This function satisfies the properties of Proposition 1. In fact, using (14) we

obtain

$$\begin{aligned}
& \int_{\mathbb{R}} \Psi(u) \mathcal{M}_{\sigma,p}(u) \frac{du}{\sqrt{2\pi}} = \int_{A(\sigma,p)} \Psi(u) \mathcal{M}_{\sigma,p}(u) \frac{du}{\sqrt{2\pi}} \\
&= \lim_{t_1, t_2 \rightarrow 0} \int_{-\pi+t_1}^{-t_2} \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) \\
&\quad \times \frac{|1 - e^{i\theta} p^{-\sigma}|^2}{(-\sqrt{2\pi} \sin \theta p^{-\sigma})} \cdot \frac{(-2 \sin(\theta) p^{-\sigma})}{|1 - e^{i\theta} p^{-\sigma}|^2} \cdot \frac{d\theta}{\sqrt{2\pi}} \\
&= \lim_{t_1, t_2 \rightarrow 0} \frac{1}{\pi} \int_{-\pi+t_1}^{-t_2} \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\
&= \frac{1}{\pi} \int_0^\pi \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^\pi \Psi(-2 \log |1 - e^{i\theta} p^{-\sigma}|) d\theta \\
&= \int_{\mathcal{T}_p} \Psi(-2 \log |1 - t_p p^{-\sigma}|) d^* t_p \\
&= \int_{\mathcal{T}_p} \Psi(2\Re(g_{\sigma,p}(t_p))) d^* t_p.
\end{aligned}$$

In the case  $|\mathcal{P}| > 1$ , we construct the function  $\mathcal{M}_{\sigma,\mathcal{P}}$  by the convolution product of  $\mathcal{M}_{\sigma,\mathcal{P}'}$  and  $\mathcal{M}_{\sigma,p}$  for  $\mathcal{P} = \mathcal{P}' \cup \{p\}$  inductively, that is

$$\mathcal{M}_{\sigma,\mathcal{P}}(u) = \int_{\mathbb{R}} \mathcal{M}_{\sigma,\mathcal{P}'}(u') \mathcal{M}_{\sigma,p}(u - u') \frac{du'}{\sqrt{2\pi}}.$$

It is easy to show that this function satisfies the statements of Proposition 1.  $\square$

Secondly, for the purpose of considering  $\lim_{|\mathcal{P}| \rightarrow \infty} \mathcal{M}_{\sigma,\mathcal{P}}$ , we define the Fourier transform of  $\mathcal{M}_{\sigma,\mathcal{P}}$ .

When  $\mathcal{P} = \{p\}$ , we define  $\widetilde{\mathcal{M}}_{\sigma,p}$  by

$$\widetilde{\mathcal{M}}_{\sigma,p}(x) = \int_{\mathbb{R}} \mathcal{M}_{\sigma,p}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}},$$

where  $\psi_x(u) = e^{ixu}$ . (The Fourier transform is sometimes defined by using  $e^{-ixu}$  instead of  $e^{ixu}$ , but here we follow the notation in [5] and [7].) As Ihara and the first author discussed in p.644 of [7], we can show

$$\widetilde{\mathcal{M}}_{\sigma,p}(x) = O\left((1 + |x|)^{-1/2}\right)$$

by using the Jessen-Wintner Theorem [11]. We define  $\widetilde{\mathcal{M}}_{\sigma,\mathcal{P}}(x)$  by

$$\widetilde{\mathcal{M}}_{\sigma,\mathcal{P}}(x) = \prod_{p \in \mathcal{P}} \widetilde{\mathcal{M}}_{\sigma,p}(x).$$

Then we have

$$\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \ll \left( (1 + |x|)^{-|\mathcal{P}|/2} \right) \quad (15)$$

from the above estimate of  $\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}$ . On the other hand, we have the trivial bound

$$\left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \right| \leq 1. \quad (16)$$

We can show the following properties (a), (b) and (c). The proofs of them are also similar to [7], pp.645-646.

(a).  $\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \in L^t$  ( $t \in [1, \infty]$ ).

(b). For any subsets  $\mathcal{P}'$  and  $\mathcal{P}$  of  $\mathbb{P}(q)$  with  $\mathcal{P}' \subset \mathcal{P}$ , from (16) we can see

$$\left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \right| \leq \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}'}(x) \right|.$$

(c). Let  $y \in \mathbb{N}$ , and put  $\mathcal{P}_y = \{p \in \mathbb{P}(q) \mid p \leq y\} \subset \mathbb{P}(q)$ . We can show the existence of  $\lim_{y \rightarrow \infty} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_y}(x)$  for  $\sigma > 1/2$ . We denote it by  $\widetilde{\mathcal{M}}_{\sigma}(x)$ . For any  $a > 0$ , this convergence is uniform on  $|x| \leq a$ .

These properties yield the next proposition which is the analogue of Proposition 3.4 in [7].

**Proposition 2.** *For  $\varepsilon > 0$  and  $\sigma \geq 1/2 + \varepsilon$ , there exists*

$$\widetilde{\mathcal{M}}_{\sigma}(x) = \lim_{y \rightarrow \infty} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_y}(x),$$

*whose convergence is uniform in  $x \in \mathbb{R}$ . For each  $\sigma > 1/2$ , the above convergence is  $L^t$ -convergence and the function  $\widetilde{\mathcal{M}}_{\sigma}(x)$  belongs to  $L^t$  ( $1 \leq t \leq \infty$ ).*

By using (15) and (16), we have

$$\widetilde{\mathcal{M}}_{\sigma}(x) = O\left((1 + |x|)^{-n/2}\right) \quad (17)$$

for any  $n \in \mathbb{N}$ . We also have

$$|\widetilde{\mathcal{M}}_{\sigma}(x)| \leq 1. \quad (18)$$

Finally, we define the function  $\mathcal{M}_{\sigma}(u)$ . For any finite set  $\mathcal{P} \subset \mathbb{P}(q)$ , we have

$$\int_{\mathbb{R}} \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}} = \mathcal{M}_{\sigma, \mathcal{P}}(u).$$

This is the Fourier inverse transform. We define

$$\mathcal{M}_{\sigma}(u) = \int_{\mathbb{R}} \widetilde{\mathcal{M}}_{\sigma}(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}},$$

where we can see that the right-hand side of this equation is convergent by using (17).

**Proposition 3.** *For  $\sigma > 1/2$ , the function  $\mathcal{M}_\sigma$  satisfies following five properties.*

- $\lim_{y \rightarrow \infty} \mathcal{M}_{\sigma, \mathcal{P}_y}(u) = \mathcal{M}_\sigma(u)$  and this convergence is uniform in  $u$ .
- The function  $\mathcal{M}_\sigma(u)$  is continuous in  $u$  and non-negative.
- $\lim_{u \rightarrow \infty} \mathcal{M}_\sigma(u) = 0$ .
- The functions  $\mathcal{M}_\sigma(u)$  and  $\widetilde{\mathcal{M}}_\sigma(x)$  are Fourier duals of each other.
- $\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \frac{du}{\sqrt{2\pi}} = 1$ .

This is the analogue of Proposition 3.5 in [7] and the proof is similar.

## 4 The key lemma.

For a fixed  $\sigma > 1/2$ ,  $\tau \in \mathbb{R}$  and a finite set  $\mathcal{P} \subset \mathbb{P}(q)$ , we put

$$\Phi_{\sigma, \tau, \mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) = \sum_{p \in \mathcal{P}} (g_{\sigma, p}(t_p p^{-i\tau}) + g_{\sigma, p}(t'_p p^{-i\tau})),$$

where  $t_{\mathcal{P}} = (t_p)_{p \in \mathcal{P}}, t'_{\mathcal{P}} = (t'_p)_{p \in \mathcal{P}} \in \mathcal{T}_{\mathcal{P}}$ . From (11) we see that

$$\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) = \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma + i\tau)),$$

where  $\psi_x(u) = \exp(ixu)$ . Therefore, to prove our Theorem 1, it is important to consider two averages

$$\text{avg}_{\text{prime}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) = \lim_{\substack{q \rightarrow \infty \\ q: \text{prime} \\ m: \text{fix}}} \sum'_{f \in S_k(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}))$$

and

$$\text{avg}_{\text{power}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) = \lim_{\substack{m \rightarrow \infty \\ q: \text{fixed prime} \\ (\text{If } 1 \geq \sigma > 1/2, q \geq Q(\mu))}} \sum'_{f \in S_k(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})).$$

Our first aim in this section is to show the following

**Lemma 1.** *Let  $\mu > \nu \geq 1$  be integers with  $\mu - \nu = 2$  and  $\mathcal{P}$  be a finite subset of  $\mathbb{P}(q)$ . In the case  $2 \leq k < 12$  or  $k = 14$ , we have*

$$\begin{aligned} \text{avg}_{\text{prime}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) &= \text{avg}_{\text{power}}(\psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}) \\ &= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma, \tau, \mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^* t_{\mathcal{P}}. \end{aligned}$$

The above convergence is uniform in  $|x| \leq R$  for any  $R > 0$ .

*Proof.* Let  $1 > \varepsilon' > 0$ . Considering the Taylor expansion we find that there exist an  $M_p = M_p(\varepsilon', R) \in \mathbb{N}$  and  $d_{m_p} \in \mathbb{C}$  ( $0 \leq m_p \leq M_p$ ) such that  $\psi_x \circ g_{\sigma,p}$  can be approximated by a polynomial as

$$\left| \psi_x \circ g_{\sigma,p}(t_p) - \sum_{m_p=0}^{M_p} d_{m_p} t_p^{m_p} \right| < \varepsilon', \quad (19)$$

uniformly on  $\mathcal{T}$  with respect to  $t_p$  and also on  $|x| \leq R$  with respect to  $x$ . Replacing  $t_p$  by  $t_p p^{-i\tau}$ , we have

$$\left| \psi_x \circ g_{\sigma,p}(t_p p^{-i\tau}) - \sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p} \right| < \varepsilon',$$

where  $c_{m_p} = d_{m_p} p^{-i\tau m_p}$ . Write

$$\Psi_{\sigma,\tau,p}(t_p; M_p) = \sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p}$$

and define

$$\Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}}) = \prod_{p \in \mathcal{P}} \Psi_{\sigma,\tau,p}(t_p; M_p) \Psi_{\sigma,\tau,p}(t'_p; M_p),$$

where  $M_{\mathcal{P}} = (M_p)_{p \in \mathcal{P}}$ .

Let  $\varepsilon'' > 0$ . Choosing  $\varepsilon'$  (depending on  $|\mathcal{P}|$  and  $\varepsilon''$ ) sufficiently small, we obtain

$$|\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) - \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}})| < \varepsilon'', \quad (20)$$

again uniformly on  $\mathcal{T}$  with respect to  $t_p$  and also on  $|x| \leq R$  with respect to  $x$ . In fact, since

$$\begin{aligned} & \psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}) \\ &= \prod_{p \in \mathcal{P}} \psi_x(g_{\sigma,p}(t_p p^{-i\tau})) \psi_x(g_{\sigma,p}(t'_p p^{-i\tau})) \\ &= \prod_{p \in \mathcal{P}} (\Psi_{\sigma,\tau,p}(t_p; M_p) + O(\varepsilon')) (\Psi_{\sigma,\tau,p}(t'_p; M_p) + O(\varepsilon')) \\ &= \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t'_{\mathcal{P}}; M_{\mathcal{P}}) + (\text{remainder terms}), \end{aligned}$$

we obtain (20).

The first step of the proof of the lemma is to express the average of the value of  $\psi_x \circ \Phi_{\sigma,\tau,\mathcal{P}}$  by using  $\Psi_{\sigma,\tau,\mathcal{P}}$ . Let  $\varepsilon > 0$ . From (20) with  $t_{\mathcal{P}} = \alpha_f^{\mu}(\mathcal{P})$ ,

$t'_{\mathcal{P}} = \beta_f^\mu(\mathcal{P})$ ,  $\varepsilon'' = \varepsilon/2$  and (10), we have

$$\begin{aligned} & \left| \sum_{f \in S_k(q^m)}' \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) - \sum_{f \in S_k(q^m)}' \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \right| \\ & < \sum_{f \in S_k(q^m)}' \varepsilon'' = \frac{\varepsilon}{2}(1 + O(E(q^m))). \end{aligned} \quad (21)$$

Therefore, if  $q^m$  is sufficiently large, from (9) we see that

$$\left| \sum_{f \in S(q^m)}' \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P})) - \sum_{f \in S(q^m)}' \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \right| < \varepsilon. \quad (22)$$

As the second step, we calculate  $\Psi_{\sigma, \tau, \mathcal{P}}$  as follows;

$$\begin{aligned} & \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\ &= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} \right) \left( \sum_{n_p=0}^{M_p} c_{n_p} e^{-\mu i n_p \theta_f(p)} \right) \\ &= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 \right. \\ & \quad + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p < n_p}}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} c_{n_p} e^{-\mu i n_p \theta_f(p)} \\ & \quad \left. + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p > n_p}}^{M_p} c_{m_p} e^{\mu i m_p \theta_f(p)} c_{n_p} e^{-\mu i n_p \theta_f(p)} \right) \\ &= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 \right. \\ & \quad \left. + \sum_{m_p=0}^{M_p} \sum_{\substack{n_p=0 \\ m_p < n_p}}^{M_p} c_{m_p} c_{n_p} (e^{\mu i (m_p - n_p) \theta_f(p)} + e^{\mu i (n_p - m_p) \theta_f(p)}) \right). \end{aligned}$$

We put  $n_p - m_p = r_p$ . Using (1), we see that

$$\begin{aligned}
& \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\
&= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (e^{\mu i r_p \theta_f(p)} + e^{-\mu i r_p \theta_f(p)}) \right) \\
&= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} \left( (e^{\mu i r_p \theta_f(p)} \right. \right. \\
&\quad \left. \left. + \sum_{\ell=1}^{\mu r_p-1} e^{i(\mu r_p-2\ell)\theta_f(p)} + e^{-\mu i r_p \theta_f(p)} \right) - \sum_{\ell=1}^{\mu r_p-1} e^{i(\mu r_p-2\ell)\theta_f(p)} \right) \\
&= \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (\lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p-2})) \right).
\end{aligned}$$

Since  $\mu = \nu + 2 \geq 3$ , by using (7), we obtain

$$\begin{aligned}
& \sum'_{f \in S(q^m)} \Psi_{\sigma, \tau, \mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}}) \\
&= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p}^2 + \sum_{r_p=1}^{M_p} \sum_{n_p=r_p}^{M_p} c_{n_p-r_p} c_{n_p} (\lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p-2})) \right) \\
&= \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 + O(E(q^m)),
\end{aligned}$$

where the implied constant of the error term depends on  $\mathcal{P}$ ,  $\mu$  and  $M_{\mathcal{P}} = M_{\mathcal{P}}(\varepsilon', R)$  (hence depends on  $\varepsilon$  under the above choice of  $\varepsilon'$ ). But still, this error term can be smaller than  $\varepsilon$  for sufficiently large  $q^m$ . Combining this with (22), we obtain

$$\left| \sum'_{f \in S(q^m)} \psi_x \circ \Phi_{\sigma, \tau, \mathcal{P}}(\alpha_f(\mathcal{P})^\mu, \beta_f(\mathcal{P})^\mu) - \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 \right| < 2\varepsilon. \quad (23)$$

As the final step, we calculate the integral in the statement of Lemma 1. For

any  $\varepsilon > 0$ , using (20), we have

$$\begin{aligned}
& \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^* t_{\mathcal{P}} \\
&= \int_{\mathcal{T}_{\mathcal{P}}} (\psi_x(\Phi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) - \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}}) + \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}})) d^* t_{\mathcal{P}} \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \Psi_{\sigma,\tau,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1}; M_{\mathcal{P}}) d^* t_{\mathcal{P}} + O(\varepsilon) \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \prod_{p \in \mathcal{P}} \left( \sum_{m_p=0}^{M_p} c_{m_p} t_p^{m_p} \right) \left( \sum_{n_p=0}^{M_p} c_{n_p} t_p^{-n_p} \right) d^* t_{\mathcal{P}} + O(\varepsilon) \\
&= \prod_{p \in \mathcal{P}} \sum_{m_p=0}^{M_p} c_{m_p}^2 + O(\varepsilon). \tag{24}
\end{aligned}$$

From (23) and (24) we find that the identity in the statement of Lemma 1 holds with the error  $O(\varepsilon)$ , but this error can be arbitrarily small, so the assertion of Lemma 1 follows.  $\square$

**Remark 5.** In the above proof, the function  $\Psi_{\sigma,\tau,\mathcal{P}}(\alpha_f^\mu(\mathcal{P}), \beta_f^\mu(\mathcal{P}); M_{\mathcal{P}})$  is expressed by

$$e^{\mu i m_p \theta_f(p)} e^{-\mu i n_p \theta_f(p)}, \quad (m_p, n_p \geq 0, \mu \geq 3).$$

When  $m_p \neq n_p$  these are written by  $\lambda_f(p^{\mu r_p})$  and  $\lambda_f(p^{\mu r_p - 2})$  ( $r_p \geq 1$ ) and, as shown above, they are included in the error terms by (7). If we try to study averages of  $\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$  itself (without considering the difference) by the same method as in this paper, we have to handle the terms of the form

$$e^{i m_p \theta_f(p)} e^{-i n_p \theta_f(p)} \quad (m_p - n_p = \pm 2).$$

However, these terms produce other “main” terms by (10), since  $\alpha_f^2(p) + \beta_f^2(p) = \lambda_f(p^2) - 1$ . This invalidates the above argument, so our method, as it is, cannot be applied to  $\log L(\text{Sym}_f^\gamma, s)$ .

When  $\tau = 0$ , Proposition 1 and Lemma 1 imply

$$\begin{aligned}
& \text{avg}_{\text{prime}}(\psi_x \circ \Phi_{\sigma,0,\mathcal{P}}) = \text{avg}_{\text{power}}(\psi_x \circ \Phi_{\sigma,0,\mathcal{P}}) \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(\Phi_{\sigma,0,\mathcal{P}}(t_{\mathcal{P}}, t_{\mathcal{P}}^{-1})) d^* t_{\mathcal{P}} \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x \left( \sum_{p \in \mathcal{P}} (g_{\sigma,p}(t_p) + g_{\sigma,p}(t_p^{-1})) \right) d^* t_{\mathcal{P}} \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(g_{\sigma,\mathcal{P}}(t_{\mathcal{P}}) + \overline{g_{\sigma,\mathcal{P}}(t_{\mathcal{P}})}) d^* t_{\mathcal{P}} \\
&= \int_{\mathcal{T}_{\mathcal{P}}} \psi_x(2\Re(g_{\sigma,\mathcal{P}}(t_{\mathcal{P}}))) d^* t_{\mathcal{P}} = \int_{\mathbb{R}} \mathcal{M}_{\sigma,\mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}}, \tag{25}
\end{aligned}$$

uniformly in  $|x| \leq R$ . This fact deduces the case  $\sigma > 1$  of the following key lemma.



**Lemma 2.** *Let  $\mu > \nu \geq 1$  be integers with  $\mu - \nu = 2$ . Suppose Assumption 1 and 2. In the case  $2 \leq k < 12$  or  $k = 14$ , for  $\sigma > 1/2$  and  $\psi_x(u) = \exp(ixu)$ , we have*

$$\begin{aligned} & \text{Avg}_{\text{prime}} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \text{Avg}_{\text{power}} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) \\ &= \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \end{aligned}$$

The above convergence is uniform in  $|x| \leq R$  for any  $R > 0$ .

We note that this lemma is actually a special case  $\Psi = \psi_x$  in our main Theorem 1. To show this lemma is the main body of the proof of Theorem 1.

*Proof in the case  $\sigma > 1$ .* Since  $\sigma > 1$ , we find a sufficiently large finite subset  $\mathcal{P} \subset \mathbb{P}(q)$  for which it holds that

$$|L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s) - L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, s)| < \varepsilon$$

and  $|\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) - \widetilde{\mathcal{M}}_\sigma(x)| < \varepsilon$  for any  $x \in \mathbb{R}$  and any  $\varepsilon > 0$ . The last inequality is provided by Proposition 2. We can choose the above  $\mathcal{P}$  which does not depend on  $q^m$ . Using this  $\mathcal{P}$ , we have

$$\begin{aligned} & \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & \leq \left| \sum'_{f \in S_k(q^m)} \left( \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \right) \right| \\ & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & \quad + \left| \int_{\mathbb{R}} \mathcal{M}_{\sigma, \mathcal{P}}(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} \right| \\ & = S_1 + S_2 + S_3, \end{aligned}$$

say. We remind the relation

$$|\psi_x(u) - \psi_x(u')| \ll |x| \cdot |u - u'| \quad (26)$$

for  $u \in \mathbb{R}$  (see Ihara [5] (6.5.19) or Ihara-Matsumoto [7]). We see that

$$S_1 \ll |x| \sum'_{f \in S_k(q^m)} \left( |\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)| \right)$$

and

$$S_3 = \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right|.$$

Therefore  $|S_1|$  and  $|S_3|$  are  $O(\varepsilon)$  for large  $|\mathcal{P}|$ , with the implied constant depending on  $R$ . As for the estimate on  $|S_2|$ , we use (25), whose convergence is uniform on  $|x| \leq R$ . This completes the proof.  $\square$

In the next two sections we will give the proof of Lemma 2 when  $1 \geq \sigma > 1/2$ .

## 5 The approximation of $L_{\mathbb{P}(q)}$ under GRH.

In this section, we suppose Assumptions 1 and 2. This section is the first step of the proof of Lemma 2 for  $1 \geq \sigma > 1/2$ .

In this section, we study the approximation of  $L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)$  by  $L_{\mathcal{P}}(\text{Sym}_f^\gamma, s)$  with suitable  $\mathcal{P}$  which depends on the level of the primitive form  $f$  (see Lemma 3 below). Recall that the level of  $f$  is  $q^m$  and  $q$  is a prime number.

Let the sets  $\mathcal{P}_{\log q^m}$  and  $\mathcal{P}_{\log q^m}^+$  be defined by

$$\mathcal{P}_{\log q^m} = \{p \in \mathbb{P}(q) \mid p \leq \log q^m\}$$

and

$$\mathcal{P}_{\log q^m}^+ = \mathcal{P}_{\log q^m} \cup \{q\}.$$

Then

$$\begin{aligned} \log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) &= \log L(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s}), \\ \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s) &= \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s}) \end{aligned}$$

and

$$\begin{aligned} &\log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s) \\ &= \log L(\text{Sym}_f^\gamma, s) - \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s). \end{aligned}$$

Define

$$F(\text{Sym}_f^\gamma, s) = \frac{L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, s)}.$$

Since

$$\begin{aligned} &\log L(\text{Sym}_f^\gamma, s) \\ &= -\log(1 - \lambda_f(q^\gamma)q^{-s}) - \sum_{p \neq q} \sum_{h=0}^{\gamma} \log(1 - \alpha_f^{\gamma-h}(p)\beta^h(p)p^{-s}) \\ &= \sum_{\ell=1}^{\infty} \frac{\lambda_f^\ell(q^\gamma)}{\ell q^{\ell s}} + \sum_{p \neq q} \sum_{h=0}^{\gamma} \sum_{\ell=1}^{\infty} \frac{\alpha_f^{(\gamma-h)\ell}(p)\beta^{h\ell}(p)}{\ell p^{\ell s}} \end{aligned}$$

for  $\sigma > 1$ , we can write

$$\log F(\text{Sym}_f^n, s) = \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{\ell p^{\ell s}} \quad (27)$$

and

$$\frac{F'}{F}(\text{Sym}_f^\gamma, s) = - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell) \log p}{p^{\ell s}} \quad (28)$$

for  $\sigma > 1$ , where the coefficients  $c_{f,\gamma,p}$  are defined by

$$c_{f,\gamma,p}(\ell) = \sum_{h=0}^{\gamma} \alpha_f^{(\gamma-h)\ell}(p) \beta_f^h(p), \quad (29)$$

for  $p \neq q$ . By Assumptions 1 and 2, the functions  $\log L(\text{Sym}_f^\gamma, s)$  are holomorphic for  $\sigma > 1/2$ . By the argument of the proof of Lemma 3 in Duke [3], we obtain

$$\left| \frac{L'(\text{Sym}_f^\gamma, s)}{L(\text{Sym}_f^\gamma, s)} \right| \ll_{\epsilon, \gamma} \log q^m + \log(1 + |t|) \quad (30)$$

for  $1/2 + \epsilon \leq \sigma \leq 2$  ( $0 < \epsilon \leq 1$ ) from (2) and (3).

Now we restrict ourselves to the case  $\gamma = \mu, \nu$ . When  $q \geq Q(\mu)$ , we have

$$\begin{aligned} & \log L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s) + \log(1 - \lambda_f(q^\gamma)q^{-s}) \\ &= - \sum_{\substack{p \leq \log q^m \\ p \neq q}} \sum_{h=0}^{\gamma} \log(1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}) \end{aligned} \quad (31)$$

for  $\sigma > 1/2$  and  $\gamma = \mu, \nu$ . Here, on the second term of the left-hand side of the above equation, since we know  $\lambda_f(q) < d(q) = 2$  and  $|\lambda_f(q^\gamma)q^{-s}| < 2^\gamma q^{-\sigma} < 2^\mu / \sqrt{q}$  from (1), we have  $|\lambda_f(q^\gamma)q^{-s}| < 2^\mu / \sqrt{Q(\mu)} < 1$  and the above logarithm is well-defined. Differentiating the both sides of (31), we obtain

$$\begin{aligned} & \frac{L'_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)} \\ &= - \frac{\lambda_f(q^\gamma)q^{-s} \log q}{1 - \lambda_f(q^\gamma)q^{-s}} - \sum_{\substack{p \leq \log q^m \\ p \neq q}} \sum_{h=0}^{\gamma} \frac{\alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s} \log p}{1 - \alpha_f^{\gamma-h}(p) \beta_f^h(p) p^{-s}}. \end{aligned}$$

The denominator  $(1 - \lambda_f(q^\gamma)q^{-s})$  has a lower bound  $1 - 2^\mu / \sqrt{Q(\mu)} > 0$  when  $q \geq Q(\mu)$ . Therefore the first term of the right-hand side of the above equation can be estimated by  $q^{-1/2} \log q$ . Hence, when  $q \geq Q(\mu)$  and  $\sigma > 1/2$ , we have

$$\begin{aligned} \left| \frac{L'_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)}{L_{\mathcal{P}_{\log q^m}^+}(\text{Sym}_f^\gamma, s)} \right| &\ll q^{-1/2} \log q + \sum_{p \leq \log q^m} \frac{p^{-\sigma} \log p}{1 - p^{-\sigma}} \\ &\ll q^{-1/2} \log q + \sum_{p \leq \log q^m} \frac{\log p}{p^{1/2}} \\ &\ll q^{-1/2} \log q + (\log q^m)^{1/2} \\ &\ll (\log q^m)^{1/2} \end{aligned} \quad (32)$$

by partial summation and the prime number theorem. From (30) and (32), we obtain

$$\frac{F'}{F}(\text{Sym}_f^\gamma, s) \ll \log q^m + \log(1 + |t|) \quad \left(\frac{1}{2} + \varepsilon \leq \sigma \leq 2\right), \quad (33)$$

under assumptions.

Now we assume  $1/2 < \sigma \leq 1$ , and put  $\sigma = 1/2 + \delta$ ,  $0 < \delta \leq 1/2$ . The following argument is similar to the proof of Proposition 5 in Duke [3]. By Mellin's formula, we know

$$e^{-y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y^{-z} \Gamma(z) dz.$$

Therefore by (28), for  $y > 0$ , we have

$$\begin{aligned} & - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{F'}{F}(\text{Sym}_f^\gamma, u+z) x^z \Gamma(z) dz, \end{aligned}$$

where  $u > 0$  and  $x > 1$ . Now assume  $(1+\delta)/2 < u \leq 3/2$ . By shifting the path of integration to  $\Re z = (1+\delta)/2 - u$  and using (33), we have

$$\begin{aligned} & - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} \\ &= \frac{F'}{F}(\text{Sym}_f^\gamma, u) + O(x^{(1+\delta)/2-u} \log q^m). \end{aligned}$$

Integrating the above equation with respect to  $u$  from  $\sigma = 1/2 + \delta$  to  $3/2$  we obtain

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, 3/2) - \log F(\text{Sym}_f^\gamma, \sigma) = \int_{\sigma}^{3/2} \frac{F'}{F}(\text{Sym}_f^\gamma, u) du \\ &= - \int_{\sigma}^{3/2} \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \log p \sum_{\ell=1}^{\infty} \frac{c_{f,\gamma,p}(\ell)}{p^{\ell u}} e^{-p^\ell/x} du \\ &+ O\left(\int_{\sigma}^{3/2} x^{(1+\delta)/2-u} du \cdot \log q^m\right). \end{aligned}$$

Separating the term corresponding to  $\ell = 1$  on the right-hand side, we see that

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, \sigma) - \log F(\text{Sym}_f^\gamma, 3/2) \\ &= - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^{3/2}} e^{-p/x} + \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^{\sigma}} e^{-p/x} \\ &+ O_{\delta} \left( \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=2}^{\infty} \frac{1}{\ell p^{\ell\sigma}} e^{-p^\ell/x} + \frac{x^{1/2-\sigma+\delta/2} \log q^m}{\log x} \right), \quad (34) \end{aligned}$$

because from (29) we see that  $c_{f,\gamma,p}(\ell) = O(1)$ .

On the right-hand side of (34), we have

$$\begin{aligned} & \sum_{p \geq \log q^m} \sum_{\ell=2}^{\infty} \frac{1}{\ell p^{\ell\sigma}} e^{-p^\ell/x} \\ & \ll \sum_{\ell=2}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/2+\ell\delta}} < \sum_{p \geq \log q^m} \frac{1}{p^{1+2\delta}} + \sum_{\ell=3}^{\infty} \frac{1}{\ell} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/2+\ell\delta}} \\ & \ll \frac{1}{(\log q^m)^\delta} + \sum_{\ell=3}^{\infty} \frac{1}{\ell^2} \sum_{p \geq \log q^m} \frac{1}{p^{\ell/3+\ell\delta}} \ll \frac{1}{(\log q^m)^\delta} \end{aligned}$$

and

$$\sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^{3/2}} e^{-p/x} \ll \sum_{p > \log q^m} \frac{1}{p^{3/2}} \ll \sum_{n > \log q^m} \frac{1}{n^{3/2}} \ll (\log q^m)^{-1/2}.$$

Moreover, using (27) we have

$$\begin{aligned} |\log F(\text{Sym}_f^\gamma, 3/2)| & \ll \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \sum_{\ell=1}^{\infty} \frac{1}{\ell p^{3\ell/2}} \\ & \ll \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \sum_{p \geq \log q^m} \frac{1}{p^{3\ell/2-\ell/4}} < \sum_{\log q^m < n} \frac{1}{n^{5/4}} \ll (\log q^m)^{-1/4}. \end{aligned}$$

Letting  $x = q^{m/4(k-1)\gamma}$  we obtain

$$\begin{aligned} & \log F(\text{Sym}_f^\gamma, \sigma) - \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{c_{f,\gamma,p}(1)}{p^\sigma} e^{-p/q^{m/4(k-1)\gamma}} \\ & = O_{\delta,k,\gamma} \left( (\log q^m)^{-\delta} + (\log q^m)^{-1/4} + (q^{m/4(k-1)\gamma})^{-\delta/2} \right). \end{aligned}$$

Since it is easy to see that  $c_{f,\gamma,p}(1) = \lambda_f(p^\gamma)$  from (1) and (29), we now obtain the following lemma.

**Lemma 3.** *Suppose Assumptions 1 and 2. Let  $Q(\mu)$  be the smallest prime number satisfying  $2^\mu / \sqrt{Q(\mu)} < 1$  and  $f$  be a primitive form in  $S_k(q^m)$ , where  $q > Q(\mu)$  is a prime. For fixed  $\gamma$  and  $\sigma = 1/2 + \delta$  ( $0 < \delta \leq 1/2$ ), we have*

$$\begin{aligned} & \log L_{\mathbb{P}(q)}(\text{Sym}_f^\gamma, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\gamma, \sigma) - \mathcal{S}_\gamma \\ & \ll (\log q^m)^{-\delta} + (\log q^m)^{-1/4} + (q^{m/4(k-1)\gamma})^{-\delta/2}, \end{aligned} \tag{35}$$

where

$$\mathcal{S}_\gamma = \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{\lambda_f(p^\gamma)}{p^\sigma} e^{-p/q^{m/(k-1)\gamma}}.$$

## 6 Proof of Lemma 2 for $1 \geq \sigma > 1/2$ .

We already proved Lemma 2 for  $\sigma > 1$  in Section 4. In this section, we prove Lemma 2 for  $1 \geq \sigma > 1/2$  by using (35) proved in the previous section, under Assumptions 1 and 2. We remind the relation

$$\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \psi_x(u) \frac{du}{\sqrt{2\pi}} = \widetilde{\mathcal{M}}_\sigma(x).$$

Our aim is to prove that

$$\left| \sum'_{f \in S_k(q^m)} \psi_\xi(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_\sigma(x) \right| \quad (36)$$

tends to 0 as  $q^m \rightarrow \infty$  for fixed  $q \geq Q(\mu)$ , when  $1 \geq \sigma > 1/2$ .

First, using (26), we can see the following inequality:

$$\begin{aligned} & \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\ & \leq \left| \sum'_{f \in S_k(q^m)} (\psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \right. \\ & \quad \left. - \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \right| \\ & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) \right| \\ & \quad + \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\ & \ll \sum'_{f \in S_k(q^m)} |x| \left( \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \sigma) - \mathcal{S}_\mu \right| + |\mathcal{S}_\mu| \right. \\ & \quad \left. + \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\nu, \sigma) - \mathcal{S}_\nu \right| + |\mathcal{S}_\nu| \right) \\ & \quad + \left| \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) - \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) \right| \\ & \quad + \left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| \\ & = \mathcal{X}_{\log q^m} + \mathcal{Y}_{\log q^m} + \mathcal{Z}_{\log q^m}, \end{aligned} \quad (37)$$

say. From Proposition 2, for any  $\varepsilon > 0$ , there exists a number  $N_0 = N_0(\varepsilon)$  for which

$$\left| \widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) - \widetilde{\mathcal{M}}_\sigma(x) \right| < \varepsilon$$

holds for any  $q^m > N_0$ , uniformly in  $x \in \mathbb{R}$ . Therefore

$$\lim_{q^m \rightarrow \infty} \mathcal{Z}_{\log q^m} = 0. \quad (38)$$

On the estimate of  $\mathcal{X}_{\log q^m}$ , by using (10) and (35), we find that

$$\begin{aligned} & \sum'_{f \in S_k(q^m)} |x| \left( \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \sigma) - \mathcal{S}_\mu \right| \right. \\ & \left. + \left| \log L_{\mathbb{P}(q)}(\text{Sym}_f^\nu, \sigma) - \log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\nu, \sigma) - \mathcal{S}_\nu \right| \right) \rightarrow 0 \end{aligned}$$

as  $q^m$  tends to  $\infty$ , uniformly in  $|x| \leq R$ . Next, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} & \sum'_{f \in S_k(q^m)} |\mathcal{S}_\gamma| \\ & \leq \left( \sum'_{f \in S_k(q^m)} 1^2 \right)^{1/2} \left( \sum'_{f \in S_k(q^m)} \left( \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{\lambda_f(p^\gamma)}{p^\sigma} e^{-p/q^{m/4(k-1)\gamma}} \right)^2 \right)^{1/2}. \end{aligned}$$

Here, the first factor is  $O(1)$  by (10), while the second factor is

$$\begin{aligned} & \ll \left( \sum'_{f \in S_k(q^m)} \sum_{p \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m}} \frac{e^{-2p/q^{m/4(k-1)\gamma}}}{p^{2\sigma}} \right. \\ & \left. + \sum'_{f \in S_k(q^m)} \sum_{\substack{p, p' \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m} \\ p > p'}} \frac{\lambda_f(p^\gamma p'^\gamma)}{(pp')^\sigma} e^{-(p+p')/q^{m/4(k-1)\gamma}} \right)^{1/2} \\ & \ll \left( \sum'_{f \in S_k(q^m)} \frac{1}{(\log q^m)^\delta} \right. \\ & \left. + \sum_{\substack{p, p' \in \mathbb{P}(q) \setminus \mathcal{P}_{\log q^m} \\ p < p'}} \frac{(pp')^{(k-1)\gamma/2} E(q^m)}{(pp')^\sigma} e^{-(p+p')/q^{m/4(k-1)\gamma}} \right)^{1/2}. \end{aligned}$$

Noting

$$e^{-p/q^{m/4(k-1)\gamma}} \ll \left( \frac{q^{m/4(k-1)\gamma}}{p} \right)^{(k-1)\gamma/2+1/2}$$

we obtain

$$\begin{aligned}
\sum'_{f \in S_k(q^m)} |\mathcal{S}_\gamma| &\ll \left( \sum'_{f \in S_k(q^m)} \frac{1}{(\log q^m)^\delta} \right. \\
&\quad \left. + E(q^m) \left( \sum_{p \in \mathbb{P}} \frac{p^{(k-1)\gamma/2}}{p^\sigma} \left( \frac{q^{m/4(k-1)\gamma}}{p} \right)^{(k-1)\gamma/2+1/2} \right)^2 \right)^{1/2} \\
&\ll \left( \frac{1}{(\log q^m)^\delta} + E(q^m) \left( \sum_{n=1}^{\infty} \frac{q^{(m/8+m/8(k-1)\gamma)}}{n^{\sigma+1/2}} \right)^2 \right)^{1/2} \\
&\ll \left( \frac{1}{(\log q^m)^\delta} + E(q^m) q^{m/2} \right)^{1/2} \\
&\ll (\log q^m)^{-\delta/2}
\end{aligned} \tag{39}$$

by (9). Hence we see that

$$\lim_{q^m \rightarrow \infty} \mathcal{X}_{\log q^m} = 0 \tag{40}$$

uniformly in  $|x| \leq R$ .

The remaining part of this section is devoted to the estimate of  $\mathcal{Y}_{\log q^m}$ . We begin with the Taylor expansion

$$\psi_x(g_{\sigma,p}(t_p)) = \exp(ixg_{\sigma,p}(t_p)) = 1 + \sum_{n=1}^{\infty} \frac{(ix)^n}{n!} g_{\sigma,p}^n(t_p),$$

where

$$\begin{aligned}
g_{\sigma,p}^n(t_p) &= (-\log(1 - t_p p^{-\sigma}))^n \\
&= \left( \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{t_p}{p^\sigma} \right)^j \right)^n \\
&= \sum_{a=1}^{\infty} \left( \sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left( \frac{t_p}{p^\sigma} \right)^a.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi_x(g_{\sigma,p}(t_p)) &= 1 + \sum_{n=1}^{\infty} \frac{(ix)^n}{n!} \sum_{a=1}^{\infty} \left( \sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left( \frac{t_p}{p^\sigma} \right)^a \\
&= 1 + \sum_{a=1}^{\infty} \sum_{n=1}^a \frac{(ix)^n}{n!} \left( \sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) \left( \frac{t_p}{p^\sigma} \right)^a,
\end{aligned}$$



which we can write as

$$\psi_x(g_{\sigma,p}(t_p)) = \sum_{a=0}^{\infty} G_a(p, x) t_p^a \quad (41)$$

with

$$G_a(p, x) = \begin{cases} 1 & a = 0, \\ \frac{1}{p^{a\sigma}} \sum_{n=1}^a \frac{(ix)^n}{n!} \left( \sum_{\substack{a=j_1+\dots+j_n \\ j_\ell \geq 1}} \frac{1}{j_1 j_2 \cdots j_n} \right) & a \geq 1. \end{cases}$$

Define

$$G_a(x) = \begin{cases} 1 & a = 0, \\ \sum_{n=1}^a \frac{x^n}{n!} \binom{a-1}{n-1} & a \geq 1. \end{cases}$$

This symbol is the same as (63) in Ihara and the first author [7]. Using this symbol we obtain

$$|G_a(p, x)| \leq \frac{1}{p^{a\sigma}} G_a(|x|) \quad (42)$$

(see (65) in [7]). We use (75), (78) and (79) in Ihara and the first author [7] below.

From (11) and (41) we find that

$$\begin{aligned} & \sum'_{f \in S_k(q^m)} \psi_x(\log L_{\mathcal{P}_{\log q^m}}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma)) \\ &= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \psi_x(g_{\sigma,p}(\alpha_f^\mu(p)) + g_{\sigma,p}(\beta_f^\mu(p))) \\ &= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \left( \sum_{a_p=0}^{\infty} G_{a_p}(p, x) \alpha_f^{\mu a_p}(p) \right) \left( \sum_{b_p=0}^{\infty} G_{b_p}(p, x) \beta_f^{\mu b_p}(p) \right) \\ &= \sum'_{f \in S_k(q^m)} \prod_{p \in \mathcal{P}_{\log q^m}} \left( \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right. \\ & \quad \left. + \sum_{\substack{0 \leq a_p, b_p \\ a_p \neq b_p}} G_{a_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\mu b_p}(p) \right) \end{aligned} \quad (43)$$

and also, from Proposition 1 and (41),

$$\begin{aligned}
\widetilde{\mathcal{M}}_{\sigma, \mathcal{P}_{\log q^m}}(x) &= \int_{\mathbb{R}} \psi_x(u) \mathcal{M}_{\sigma, \mathcal{P}_{\log q^m}}(u) \frac{du}{\sqrt{2\pi}} \\
&= \int_{\mathcal{T}_{\mathcal{P}_{\log q^m}}} \psi_x(2\Re(g_{\sigma, \mathcal{P}_{\log q^m}}(t_{\mathcal{P}_{\log q^m}}))) d^* t_{\mathcal{P}_{\log q^m}} \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \psi_x(2\Re(g_{\sigma, p}(t_p))) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \psi_x(g_{\sigma, p}(t_p)) \psi_x(g_{\sigma, p}(\overline{t_p})) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \int_{\mathcal{T}_p} \left( \sum_{a_p=0}^{\infty} G_{a_p}(p, x) t_p^{a_p} \right) \left( \sum_{b_p=0}^{\infty} G_{b_p}(p, x) t_p^{-b_p} \right) d^* t_p \\
&= \prod_{p \in \mathcal{P}_{\log q^m}} \left( \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right). \tag{44}
\end{aligned}$$

Write  $\mathcal{P}_{\log q^m} = \{p_1, p_2, \dots, p_L\}$ , where  $p_l$  means the  $l$ -th prime number. (So  $L = \pi(\log q^m)$ .) Substituting (43) and (44) into the definition of  $\mathcal{Y}_{\log q^m}$ , and noting (10), we obtain

$$\begin{aligned}
\mathcal{Y}_{\log q^m} &= \left| E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \right. \\
&\quad + \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left( \sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \\
&\quad \times \left( \sum_{\substack{0 \leq a_{p_\ell}, b_{p_\ell} \\ a_{p_\ell} \neq b_{p_\ell}}} G_{a_{p_\ell}}(p_\ell, x) G_{b_{p_\ell}}(p_\ell, x) \alpha_f^{\mu_{a_{p_\ell}}}(p_\ell) \beta_f^{\mu_{b_{p_\ell}}}(p_\ell) \right)^{j_\ell} \Big|.
\end{aligned}$$

If we see that

$$\begin{aligned}
\mathcal{Y}'_{\log q^m} &= \left| \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left( \sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\
&\quad \times \left( \sum_{\substack{0 \leq a_{p_\ell}, b_{p_\ell} \\ a_{p_\ell} \neq b_{p_\ell}}} G_{a_{p_\ell}}(p_\ell, x) G_{b_{p_\ell}}(p_\ell, x) \alpha_f^{\mu_{a_{p_\ell}}}(p_\ell) \beta_f^{\mu_{b_{p_\ell}}}(p_\ell) \right)^{j_\ell} \Big| \tag{45}
\end{aligned}$$

and

$$\mathcal{Y}''_{\log q^m} = E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \tag{46}$$

tend to 0, then we obtain that  $\mathcal{Y}_{\mathcal{P}_{\log q^m}}$  tends to 0 as  $q^m$  tends to  $\infty$ .

We first consider the second inner sum on the right-hand side of (45). Letting  $a_p - b_p = r_p$  for the part of  $a_p > b_p$  and letting  $b_p - a_p = r_p$  for the part of  $b_p > a_p$ , we obtain

$$\begin{aligned}
& \sum_{\substack{0 \leq a_p, b_p \\ a_p \neq b_p}} G_{a_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\mu b_p}(p) \\
&= \sum_{0 \leq a_p < b_p} + \sum_{a_p > b_p \geq 0} \\
&= \sum_{r_p \geq 1} \sum_{b_p \geq r_p} G_{b_p - r_p}(p, x) G_{b_p}(p, x) \alpha_f^{\mu(b_p - r_p)}(p) \beta_f^{\mu b_p}(p) \\
&\quad + \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \alpha_f^{\mu a_p}(p) \beta_f^{\mu(a_p - r_p)}(p) \\
&= \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \left( \beta_f^{\mu r_p}(p) + \alpha_f^{\mu r_p}(p) \right) \\
&= \sum_{r_p \geq 1} \sum_{a_p \geq r_p} G_{a_p}(p, x) G_{a_p - r_p}(p, x) \left( \lambda_f(p^{\mu r_p}) - \lambda_f(p^{\mu r_p - 2}) \right),
\end{aligned}$$

where the last equation is deduced by

$$\begin{aligned}
\lambda_f(p^{\mu r_p}) &= \sum_{h=0}^{\mu r_p} \alpha_f^{\mu r_p - h}(p) \beta_f^h(p) \\
&= \alpha_f^{\mu r_p} + \alpha_f^{\mu r_p - 2} + \alpha_f^{\mu r_p - 4} + \dots + \beta_f^{\mu r_p - 4} + \beta_f^{\mu r_p - 2} + \beta_f^{\mu r_p}
\end{aligned}$$

which is from (1). Letting

$$G_{p,x}(r) = \sum_{a_p \geq r} G_{a_p}(p, x) G_{a_p - r}(p, x),$$

from (45) we obtain

$$\begin{aligned}
\mathcal{Y}'_{\mathcal{P}_{\log q^m}} &= \left| \sum'_{f \in S_k(q^m)} \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left( \sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\
&\quad \times \left. \left( \sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{\mu r_{p_\ell}}) - \lambda_f(p_\ell^{\mu r_{p_\ell} - 2})) \right)^{j_\ell} \right| \\
&= \left| \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left( \sum_{a_{p_\ell}=0}^{\infty} G_{a_{p_\ell}}^2(p_\ell, x) \right)^{1-j_\ell} \right. \\
&\quad \times \left. \sum'_{f \in S_k(q^m)} \prod_{\ell=1}^L \left( \sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{\mu r_{p_\ell}}) - \lambda_f(p_\ell^{\mu r_{p_\ell} - 2})) \right)^{j_\ell} \right|.
\end{aligned} \tag{47}$$

**Remark 6.** Here we remark why we only consider the case  $\mu - \nu = 2$  in the present paper. If we consider the case that  $\nu$  has the same parity with  $\mu$  but  $\mu - \nu = 2h > 2$ , and discuss analogously as above, then the factor of the form

$$\prod_{h=0}^{(\mu-\nu)/2-1} \prod_{\ell=1}^L \left( \sum_{r_{p_\ell} \geq 1} G_{a_{p_\ell}, x}(r_{p_\ell}) (\lambda_f(p_\ell^{(\mu-2h)r_{p_\ell}}) - \lambda_f(p_\ell^{(\mu-2h)r_{p_\ell}-2})) \right)^{j_\ell}$$

appears. The summation of this factor over primitive forms cannot be included in the error term, because of (7).

Let us continue the argument. From (47) we obtain

$$\mathcal{Y}'_{\mathcal{P}_{\log q^m}} \leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L |G_{p_\ell, x}(0)|^{1-j_\ell} \left| \sum'_{f \in S_k(q^m)} \sum_{1 \leq n} \mathcal{G}_x(n) \lambda_f(n) \right|,$$

where

$$\mathcal{G}_x(n) = \begin{cases} \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} (-1)^{r''(p_\ell)} G_{p_\ell, x}(r_{p_\ell}) & n = \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} p_\ell^{r'_{p_\ell}}, \\ 0 & \text{otherwise,} \end{cases}$$

$r'_{p_\ell} = \mu r_{p_\ell}$  or  $\mu r_{p_\ell} - 2$ , and

$$r''(p_\ell) = \begin{cases} 0 & r'_{p_\ell} = \mu r_{p_\ell} \\ 1 & r'_{p_\ell} = \mu r_{p_\ell} - 2. \end{cases}$$

We divide the summation of  $\mathcal{G}_x(n) \lambda_f(n)$  above into two parts according to the conditions  $n \leq M$  and  $n > M$ , where  $M$  is a suitable constant depending on  $k$ ,  $\log q^m$  and  $p_\ell$  ( $1 \leq \ell \leq L$ ) defined below. We apply the formula (7) with  $n \neq 1$  to the summation of  $n \leq M$ . And we use  $\lambda_f(n) \ll n^\eta$  (where  $\eta$  is an arbitrarily small positive number which will be specified later) by the Ramanujan-Petersson estimate for the estimation of summation of  $n > M$ . We obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &\leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L |G_{p_\ell, x}(0)|^{1-j_\ell} \\ &\quad \times \left( E(q^m) \sum_{1 \leq n \leq M} |\mathcal{G}_x(n)| n^{(k-1)/2} + \sum_{n > M} |\mathcal{G}_x(n)| n^\eta \right). \end{aligned} \quad (48)$$

From (75), (78) and (79) in [7] and (42) in this paper, we see that

$$\begin{aligned} |G_{p_\ell, x}(0)| &\leq \sum_{a_{p_\ell}=0}^{\infty} |G_{a_{p_\ell}}(p_\ell, x)|^2 \leq \sum_{a_{p_\ell}=0}^{\infty} \frac{1}{p_\ell^{2a_{p_\ell}\sigma}} G_{a_{p_\ell}}^2(|x|) \\ &\leq \left( \sum_{a_{p_\ell}=0}^{\infty} \frac{1}{p_\ell^{a_{p_\ell}\sigma}} G_{a_{p_\ell}}(|x|) \right)^2 = \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \end{aligned} \quad (49)$$

and

$$\begin{aligned}
|G_{p_\ell, x}(r_{p_\ell})| &\leq \sum_{a_{p_\ell} \geq r_{p_\ell}} |G_{a_{p_\ell}}(p_\ell, x) G_{a_{p_\ell} - r_{p_\ell}}(p_\ell, x)| \\
&\leq \sum_{a_{p_\ell} \geq r_{p_\ell}} \frac{1}{p_\ell^{a_{p_\ell} \sigma} p_\ell^{(a_{p_\ell} - r_{p_\ell}) \sigma}} G_{a_{p_\ell}}(|x|) G_{a_{p_\ell} - r_{p_\ell}}(|x|) \\
&\leq \sum_{a'_{p_\ell} = 0}^{\infty} \frac{1}{p_\ell^{(a'_{p_\ell} + r_{p_\ell}) \sigma} p_\ell^{a'_{p_\ell} \sigma}} G_{a'_{p_\ell} + r_{p_\ell}}(|x|) G_{a'_{p_\ell}}(|x|) \\
&\leq \frac{1}{p^{r_{p_\ell} \sigma}} \sum_{a'_{p_\ell} = 0}^{\infty} \frac{1}{p_\ell^{a'_{p_\ell} \sigma} p_\ell^{a'_{p_\ell} \sigma}} G_{a'_{p_\ell}}(|x|) G_{a'_{p_\ell}}(|x|) L_{r_{p_\ell}}(|x|) \\
&\leq \frac{L_{r_{p_\ell}}(|x|)}{p_\ell^{r_{p_\ell} \sigma}} \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \\
&\leq \frac{1}{p^{r_{p_\ell} \sigma/2}} \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left( \exp \left( \frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right), \tag{50}
\end{aligned}$$

where

$$L_r(x) = \sum_{m=0}^r G_m(x)$$

(the same as [74] in [7]). Therefore, when

$$n = \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} p_\ell^{r'_{p_\ell}}$$

with  $r'_{p_\ell} = \mu r_{p_\ell}$  or  $\mu r_{p_\ell} - 1$  for  $r_{p_\ell} \geq 1$ , from (50) we have

$$\begin{aligned}
\mathcal{G}_x(n) &\leq \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} |G_{p_\ell, x}(r_{p_\ell})| \\
&\leq \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} \frac{1}{p_\ell^{r_{p_\ell} \sigma/2}} \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left( \exp \left( \frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right) \\
&\leq \frac{1}{n^{\sigma/2\mu}} \prod_{\substack{1 \leq \ell \leq L \\ j_\ell = 1}} \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left( \exp \left( \frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right). \tag{51}
\end{aligned}$$

From (48), (49) and (51), we obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &\ll \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\mu \in \{0, 1\}}} \prod_{\ell=1}^L \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left( \exp \left( \frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right)^{j_\ell} \\ &\times \left( E(q^m) \sum_{\substack{n \leq M \\ p \nmid n \text{ for } p > \log q^m}} \frac{n^{(k-1)/2}}{n^{\sigma/2\mu}} + \sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{n^\eta}{n^{\sigma/2\mu}} \right). \end{aligned}$$

Here we choose  $\eta = \sigma/4\mu$ . Then the second inner sum is

$$\begin{aligned} &\sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{1}{n^{\sigma/4\mu}} < \frac{1}{M^{\sigma/8\mu}} \sum_{\substack{n > M \\ p \nmid n \text{ for } p > \log q^m}} \frac{1}{n^{\sigma/8\mu}} \\ &< \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{1}{1 - p_l^{-\sigma/8\mu}} = \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{p_l^{\sigma/8\mu}}{p_l^{\sigma/8\mu} - 1} \\ &\ll \frac{1}{M^{\sigma/8\mu}} \prod_{l=1}^L \frac{p_l^{\sigma/8\mu}}{p_l^{\sigma/8\mu} - p_l^{\sigma/8\mu}/2} \ll \frac{2^L}{M^{\sigma/8\mu}}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \mathcal{Y}'_{\mathcal{P}_{\log q^m}} &\leq \sum_{\substack{(j_1, \dots, j_L) \neq (0, \dots, 0) \\ j_\ell \in \{0, 1\}}} \prod_{\ell=1}^L \left( \exp \left( \frac{|x|}{p_\ell^\sigma - 1} \right) \right)^2 \left( \exp \left( \frac{|x|}{p_\ell^{\sigma/2} - 1} \right) \right)^{j_\ell} \\ &\times \left( E(q^m) M^{(k+1)/2 - \sigma/2\mu} + \frac{2^L}{M^{\sigma/8\mu}} \right) \\ &\leq 2^L \prod_{\ell=1}^L \left( \exp \left( \frac{|x|}{p_\ell^{1/4} - 1} \right) \right)^3 \\ &\times \left( E(q^m) M^{(k+1)/2 - 1/4\mu} + \frac{2^L}{M^{1/16\mu}} \right), \end{aligned}$$

since  $\sigma > 1/2$ . For large  $\ell$ , we know

$$\exp \left( \frac{|x|}{p_\ell^{1/4} - 1} \right) \leq \exp \left( \frac{R}{p_\ell^{1/4} - 1} \right) < 2 \quad (52)$$

for  $|x| \leq R$ , hence

$$\prod_{\ell=1}^L \left( \exp \left( \frac{|x|}{p_\ell^{1/4} - 1} \right) \right)^3 \ll_R 2^{3L}.$$

Therefore we obtain

$$\mathcal{Y}'_{\mathcal{P}_{\log q^m}} \leq 2^{4L} \left( E(q^m) M^{c(k, \mu)} + \frac{2^L}{M^{1/16\mu}} \right), \quad (53)$$

where  $c(k, \mu) = (k+1)/2 - 1/4\mu$ . Noting the fact that the number of the prime numbers less than  $2^5 = 32$  is 11, we choose

$$M = \left( \frac{p_1 \cdots p_L}{(2^5)^{L-11}} \right)^{1/c(k, \mu)}.$$

Since  $c(k, \mu) \leq 15/2 < 8$ , we see that

$$\begin{aligned} M &= (p_1 \cdots p_{11})^{1/c(k, \mu)} \left( \frac{p_{12}}{2^5} \cdot \frac{p_{13}}{2^5} \cdot \frac{p_L}{2^5} \right)^{1/c(k, \mu)} \\ &> (p_1 \cdots p_{11})^{2/15} > (200560490130)^{1/8} > 25. \end{aligned}$$

For large  $m$  or  $q$ , by the prime number theorem we have

$$\begin{aligned} p_1 \cdots p_L &= \exp(\log p_1 + \cdots + \log p_L) \\ &= \exp(\log q^m + O((\log q^m) \exp(-c_1 \sqrt{\log \log q^m}))) \\ &\leq q^{m(1+c_2 \exp(-c_1 \sqrt{\log \log q^m}))} \end{aligned}$$

where  $c_1, c_2$  are positive constants. By using (9), we have

$$\begin{aligned} 2^{4L} E(q^m) M^{c(k, \mu)} &\leq 2^{4L} E(q^m) \cdot \frac{p_1 \cdots p_L}{2^{5L-55}} \\ &\ll \frac{p_1 \cdots p_L}{2^L} \cdot E(q^m) \ll \frac{q^{m(1+c_2 \exp(-c_1 \sqrt{\log \log q^m}))}}{2^L} \cdot q^{-m}. \end{aligned} \quad (54)$$

Again by the prime number theorem, we see that

$$2^L = 2^{\log q^m (1+o(1)) / \log \log q^m} = (q^m)^{\log 2(1+o(1)) / \log \log q^m},$$

so we find that the right-hand side of (54) is

$$\ll (q^m)^{c_2 \exp(-c_1 \sqrt{\log \log q^m}) - \log 2(1+o(1)) / \log \log q^m},$$

whose exponent is negative for large  $q^m$ . Therefore this tends to 0 as  $q^m$  tends to  $\infty$ .

Next, we have

$$\begin{aligned} \frac{2^{4L} 2^L}{M^{1/16\mu}} &= 2^{5L} \left( \frac{p_1 \cdots p_L}{2^{5L-55}} \right)^{-1/16\mu c(k, \mu)} \\ &\ll \frac{(2^L)^{5+5/16\mu c(k, \mu)}}{(p_1 \cdots p_L)^{1/16\mu c(k, \mu)}} \\ &\ll \left( \frac{(2^{80\mu c(k, \mu)+5})^L}{p_1 \cdots p_L} \right)^{1/16\mu c(k, \mu)}, \end{aligned}$$

so, putting  $d(k, \mu) = 2^{80\mu c(k, \mu)+5}$ , the above is

$$\begin{aligned} &= \left( \frac{d(k, \mu)}{p_1} \cdots \frac{d(k, \mu)}{p_{\pi(d(k, \mu))}} \frac{(d(k, \mu))^{L-\pi(d(k, \mu))}}{p_{\pi(d(k, \mu))+1} \cdots p_L} \right)^{1/16\mu c(k, \mu)} \\ &\ll_k \left( \frac{d(k, \mu)}{p_{\pi(d(k, \mu))+1}} \right)^{(L-\pi(d(k, \mu)))/16\mu c(k, \mu)}. \end{aligned} \quad (55)$$

Since the quantity in the parentheses is smaller than 1, we find that this also tends to 0 as  $q^m$  tends to  $\infty$ . Therefore from (53) we conclude that  $\mathcal{Y}'_{\mathcal{P}_{\log q^m}}$  tends to 0 as  $q^m$  tends to  $\infty$ .

The idea of evaluating  $\mathcal{Y}''_{\mathcal{P}_{\log q^m}}$ , defined by (46), is essentially similar, but much simpler. First, using (49), we have

$$\begin{aligned}\mathcal{Y}''_{\mathcal{P}_{\log q^m}} &= E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \sum_{a_p=0}^{\infty} G_{a_p}^2(p, x) \leq E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} |G_{p,x}(0)| \\ &\leq E(q^m) \prod_{p \in \mathcal{P}_{\log q^m}} \left( \exp \left( \frac{|x|}{p^\sigma - 1} \right) \right)^2.\end{aligned}$$

Then, by an argument similar to (52), the above is

$$\ll_R E(q^m) 2^L \ll E(q^m) e^{\log q^m / \log \log q^m} = E(q^m) (q^m)^{(1+o(1)) / \log \log q^m},$$

which tends to 0 as  $q^m$  tends to  $\infty$ . Therefore we now arrive at the assertion

$$\lim_{q^m \rightarrow \infty} \mathcal{Y}_{\mathcal{P}_{\log q^m}} = 0. \quad (56)$$

Finally we see that Lemma 2 is established, by substituting (38), (40) and (56) into (37).

## 7 Completion of the proof of Theorem 1.

The only remaining task now is to deduce the general statement of our Theorem 1 from Lemma 2. This can be done by using the general principle on the weak convergence of probability measures (as indicated in Remark 3.2 of [9]), but here we follow a more self-contained treatment given in Ihara and the first author [7]. In this section, we just explain the outline of the proof of Theorem 1, because the argument is the same as that in [7].

For any  $\varepsilon > 0$ , the aim of this section is to prove that

$$\sum'_{f \in S(q^m)} \Psi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) \quad (57)$$

tends to

$$\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}$$

as  $q^m$  tends to  $\infty$ .

We define the set  $\Lambda$  of the function  $\Phi$  on  $\mathbb{C}$  by

$$\Lambda = \{\Phi \in L^1 \cap L^\infty \mid \Phi^\vee \in L^1 \cap L^\infty, (\Phi^\vee)^\wedge = \Phi\},$$

where  $\Phi^\wedge$  means the Fourier transform of  $\Phi$  and  $\Phi^\vee$  means the Fourier inverse transform of  $\Phi$ . We know

$$\Phi(u) = \int_{\mathbb{R}} \Phi^\wedge(x) \psi_{-u}(x) \frac{dx}{\sqrt{2\pi}} = \int_{\mathbb{R}} \Phi^\wedge(x) \psi_{-x}(u) \frac{dx}{\sqrt{2\pi}}.$$



Since we also know  $\widetilde{\mathcal{M}}_\sigma \in \Lambda$  from Proposition 2 and Proposition 3, we have  $\mathcal{M}_\sigma \in \Lambda$ . Therefore we have

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Phi(u) \frac{du}{\sqrt{2\pi}} &= \int_{\mathbb{R}} \overline{\mathcal{M}_\sigma}^\wedge(x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(-x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}}. \end{aligned}$$

In the case of  $\Psi = \Phi \in \Lambda$ , we can see

$$\begin{aligned} & \left| \sum'_{f \in S(q^m)} \Phi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) - \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Phi(u) \frac{du}{\sqrt{2\pi}} \right| \\ &= \left| \sum'_{f \in S_k(q^m)} \int_{\mathbb{R}} (\Phi^\wedge(x) \psi_{-x}(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \frac{dx}{\sqrt{2\pi}} \right. \\ & \quad \left. - \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(-x) \Phi^\wedge(x) \frac{dx}{\sqrt{2\pi}} \right| \\ &= \left| \sum'_{f \in S_k(q^m)} \int_{\mathbb{R}} (\Phi^\wedge(-x) \psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) \frac{dx}{\sqrt{2\pi}} \right. \\ & \quad \left. - \int_{\mathbb{R}} \widetilde{\mathcal{M}}_\sigma(x) \Phi^\wedge(-x) \frac{dx}{\sqrt{2\pi}} \right| \\ &\leq \int_{\mathbb{R}} |\Phi^\wedge(-x)| \\ & \quad \times \left| \sum'_{f \in S_k(q^m)} (\psi_x(\log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma))) - \widetilde{\mathcal{M}}_\sigma(x) \right| \frac{dx}{\sqrt{2\pi}}. \end{aligned}$$

We divide the above integral into two parts,  $|x| \leq R$  and  $|x| > R$  by sufficiently large  $R$ . Since  $\psi_x$  and  $\widetilde{\mathcal{M}}_\sigma(x)$  are bounded (see (18)) and  $\Phi^\wedge \in L^1$ , the integral on  $|x| > R$  is small for sufficiently large  $R$ . The other integral on  $|x| \leq R$  is then also small by Lemma 2, for large  $q$  or  $m$ . Therefore the desired assertion holds for  $\Psi \in \Lambda$ .

In the case that  $\Psi$  is a compactly supported function on  $C^\infty$ , this is a element in the Schwartz space. Therefore  $\Psi \in \Lambda$ .

In the case that  $\Psi$  is a compactly supported continuous function, this is approximated by compactly supported functions in  $C^\infty$ . Therefore, in this case Theorem 1 is established. Especially, in the case that  $\Psi$  is a characteristic function on a compact subset,  $\Psi$  is approximated by compactly supported continuous function. Therefore, in this case the proof is complete.

Finally, we consider the case that  $\Psi$  is a bounded continuous function. For any  $R > 0$ , there exists a compactly supported continuous function  $\Psi_R$  such that  $\Psi_R(x) = \Psi(x)$  for  $|x| \leq R$  and  $|\Psi_R(x)| \leq |\Psi(x)|$  for  $|x| > R$ . We already

know that

$$\lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi_R \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) = \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}},$$

where the above equation is proved for  $q \geq Q(\mu)$  when  $1 \geq \sigma > 1/2$  in the case of  $m \rightarrow \infty$ . For the right-hand side of this equation, we have

$$\begin{aligned} & \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x| > R} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} + \int_{|x| \leq R} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x| > R} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} + \int_{|x| \leq R} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}} \\ &= \int_{|x| > R} \mathcal{M}_\sigma(u) (\Psi_R(u) - \Psi(u)) \frac{du}{\sqrt{2\pi}} + \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned}$$

As for the former integral, we remind that  $\mathcal{M}_\sigma$  is non-negative to find

$$\int_{|x| > R} \mathcal{M}_\sigma(u) (\Psi_R(u) - \Psi(u)) \frac{du}{\sqrt{2\pi}} \ll \int_{|x| > R} \mathcal{M}_\sigma(u) \frac{du}{\sqrt{2\pi}}$$

which tends to 0 as  $R$  tends to  $\infty$ , since we know

$$\int_{\mathbb{R}} \mathcal{M}_\sigma(u) \frac{du}{\sqrt{2\pi}} = 1$$

by Proposition 3. Hence we have

$$\begin{aligned} & \lim_{R \rightarrow \infty} \lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi_R \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) \\ &= \lim_{R \rightarrow \infty} \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi_R(u) \frac{du}{\sqrt{2\pi}} = \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}}. \end{aligned} \quad (58)$$

Finally, by using the same argument as in p.675 in Ihara and the first author [7], from (58) we obtain

$$\lim_{\substack{q \rightarrow \infty \\ \text{or } m \rightarrow \infty}} \sum'_{f \in S(q^m)} \Psi \circ \log L_{\mathbb{P}(q)}(\text{Sym}_f^\mu, \text{Sym}_f^\nu, \sigma) = \int_{\mathbb{R}} \mathcal{M}_\sigma(u) \Psi(u) \frac{du}{\sqrt{2\pi}},$$

which is the conclusion of our Theorem 1.

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Kohji Matsumoto:

Graduate School of Mathematics, Nagoya University, Furocho, Chikusa-ku,  
Nagoya 464-8602, Japan.  
kohjimat@math.nagoya-u.ac.jp

Yumiko Umegaki:

Department of Mathematical and Physical Sciences, Nara Women's University,

Kitaouya Nishimachi, Nara 630-8506, Japan.  
ichihara@cc.nara-wu.ac.jp